MATH 245 S18, Exam 3 Solutions

1. Carefully define the following terms: $\cap$, $\cup$, (absolute) complement, Cartesian product.

Given sets S,T, $S\cap T=\{x : x\in S \wedge x\in T\}$. Given sets S,T, $S\cup T=\{x : x\in S\vee x\in T\}$. Given sets S,U with $S\subseteq U$, we define the absolute complement of S as $U\setminus S$. Given sets S,T, we define the Cartesian product of S,T as $\{(x,y) : x\in S, y\in T\}$.

2. Carefully define the following terms: relation, symmetric (relation), antisymmetric (relation), trichotomous (relation).

Given sets S,T, a relation from S to T is a subset of $S\times T$. A relation R on S is symmetric if for all $x,y\in S$, $xRy\to yRx$. A relation R on S is antisymmetric if for all $x,y\in S$, $(xRy\wedge yRx)\to x=y$. A relation R on S is trichotomous if for all $x,y\in S$, $x=y\vee xRy\vee yRx$.

3. Let $S=\{a,b\}$. Give a two-element subset of $2^{S\times S}$. Be careful with notation.

Note that $S\times S=\{(a,a),(a,b),(b,a),(b,b)\}$. Elements of $2^{S\times S}$ are subsets of $S\times S$. We seek a set, which contains two elements. Each of those elements must be a subset of $S\times S$, namely a set of ordered pairs. Many solutions are possible, such as $\{\{(a,a)\},\{(b,b)\}\}$ or $\{\emptyset, S\times S\}$ or $\{\{(a,a),(a,b)\},\{(a,a),(b,a)\}\}$.

4. Let S be a set. Prove that $S\cup\emptyset=S$.

This must be proved in two parts. First we prove $\subseteq$: Let $x\in S\cup\emptyset$. Then $x\in S\vee x\in\emptyset$. We have two cases: $x\in S$ or $x\in\emptyset$. The second case can't happen, so $x\in S$. This proves $S\cup\emptyset\subseteq S$. Next, we prove $\supseteq$. Let $x\in S$. By addition, $x\in S\vee x\in\emptyset$. Hence $x\in S\cup\emptyset$. This proves $S\cup\emptyset\supseteq S$.

5. Give a partition of $\mathbb{Z}$ with three parts.

Many solutions are possible; all of them consist of a set of three parts such as $\{P_0,P_1,P_2\}$. One solution is $P_0=\{0\}, P_1=\mathbb{N}, P_2=\{x\in\mathbb{Z} : x<0\}$. Another solution is to apply the Division Algorithm with 3. P_i will be the set of integers with remainder i (which must be 0, 1, or 2). Another solution is $P_0=\{0\}, P_1=\{1\}, P_2=\mathbb{Z}\setminus\{0,1\}$.

For problems 6 and 7, take ground set $S=\{-1,0,1\}$ with relation $R=\{(a,b) : a\le b^2\}$.

6. With R,S as above, prove or disprove that R is reflexive.

The statement is true. Because $-1\le(-1)^2$, $(-1,-1)\in R$. Because $0\le 0^2$, $(0,0)\in R$. Because $1\le 1^2$, $(1,1)\in R$. These three together imply that R is reflexive.

7. With R,S as above, prove or disprove that R is transitive.

The statement is false. We need a specific counterexample. There is only one (it can be found by drawing the relation's digraph). Because $1\le(-1)^2$, $(1,-1)\in R$. Because $-1\le 0^2$, $(-1,0)\in R$. However, $(1,0)\notin R$, because $1\not\le 0^2$. Hence R is not transitive.

8. Prove or disprove: For all sets R,S, we have $R\setminus S=R\Delta S$.

The statement is false. We need a specific counterexample. Many are possible. A simple one is $R=\{1,3\}, S=\{2,3\}$. We have $R\setminus S=\{1\}$, while $R\Delta S=(R\setminus S)\cup(S\setminus R)=\{1,2\}$.

9. Prove or disprove: For all sets R,S,T satisfying $R\subseteq S$, $S\subseteq T$, and $T\subseteq R$, we must have $R=S$.

The statement is true. To prove $R=S$, we need to prove $R\subseteq S$ (one of our hypotheses already), and $S\subseteq R$. Let $x\in S$. Since $S\subseteq T$, $x\in T$. Since $T\subseteq R$, $x\in R$. Hence $S\subseteq R$.

10. Prove or disprove: $|\mathbb{N}|=|\mathbb{N}_0\times\mathbb{N}_0|$.

The statement is true.
PROOF 1: As in Thm 9.17 and Exercise 9.24, for any $n\in\mathbb{N}$ we can uniquely write $n=2^a(2b+1)$, and pair $n\leftrightarrow(a,b)$.
PROOF 2: We write all the ordered pairs in $\mathbb{N}_0\times\mathbb{N}_0$ in the first quadrant at their locations, and take a zig-zag path starting at the origin and passing through all the pairs. We pair the n^{th} position along the path with the ordered pair at that position.
PROOF 3: We pair $\mathbb{N}$ with a subset of $\mathbb{N}_0\times\mathbb{N}_0$, for example via $n\leftrightarrow(n,0)$. This proves that $|\mathbb{N}|\le|\mathbb{N}_0\times\mathbb{N}_0|$. We next pair $\mathbb{N}_0\times\mathbb{N}_0$ with a subset of $\mathbb{N}$, for example via $(a,b)\leftrightarrow 2^a3^b$. This proves that $|\mathbb{N}|\ge|\mathbb{N}_0\times\mathbb{N}_0|$. Lastly, we apply the Cantor-Schröder-Bernstein Theorem.